Regularity Structures for Interacting Photon Fields

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Abstract

We adapt Mr Martin Hairer's well written theory of regularity structures to a class of stochastic Maxwell-type PDEs modeling interacting photon fields with nonlinear coupling and additive space-time noise. We construct a vector-valued regularity structure encoding the gauge/vector-symbols needed for electromagnetic fields, identify the renormalisation group and compute the explicit counterterms required to obtain finite models. Our main result is a local (in time) well-posedness theorem for the renormalised stochastic Maxwell system in a subcritical regime (spatial dimension $d \leq 3$) and under suitable gauge-fixing / damping. We provide a proof sketch following model construction, renormalisation (BPHZ-type), and reconstruction, and give a toy numerical scheme for computing renormalisation constants on a lattice via mollifier-dependent diagram evaluation. The approach gives a principled renormalisation prescription for interacting photon SPDEs and a framework for numerical experiments.

Contents

1 Introduction

The theory of regularity structures provides a robust method to treat ill-posed stochastic PDEs exhibiting subcritical singularities. Classical applications have focussed on scalar/parabolic equations (KPZ, Φ^4). In this work we extend these ideas to Maxwell-type systems describing vector electromagnetic fields with nonlinear self-interactions and stochastic forcing. Our objectives are:

- build a regularity structure adapted to vector-valued fields with gauge structure;
- identify the renormalisation group and explicit counterterms (mass, current, gauge-fixing);
- prove local well-posedness (existence/uniqueness of modelled distributions and reconstruction) for the renormalised equation;
- outline numerical recipes to compute renormalisation constants.

To make this manuscript self-contained and avoid external cross-references, we include a short expository section below summarising the motivating background, diagrammatic bookkeeping, and examples that typically appear in the literature on regularity structures and stochastic Maxwell-type models. This summary replaces external citations and provides the reader a compact guide to the constructions used herein.

2 Expository background and motivating notes

This section presents brief, self-contained expository notes which motivate the constructions used in the rest of the manuscript. The presentation is intentionally high-level but concrete: it sketches the essential concepts of regularity structures, renormalisation via BPHZ characters, and the typical diagrammatic divergences that arise for Maxwell-like systems with vector indices.

2.1 Core ideas of regularity structures

Regularity structures separate the analysis of singular SPDEs into three layers:

1. an abstract algebraic layer (the symbol/trees space $\mathcal{T}$ and structure group $\mathcal{G}$), which organises possible local expansions of solutions;
2. analytic models $Z = (\Pi, \Gamma)$ which map abstract symbols to concrete distributions and satisfy uniform analytical bounds;
3. a fixed-point / reconstruction layer, where one solves an abstract fixed-point problem in the space of modelled distributions and then reconstructs an actual distribution from the abstract solution.

The method handles subcritical equations by truncating the symbol space to symbols with homogeneity above a chosen threshold, and then controlling the remainder via Schauder-type estimates adapted to the abstract integration operator.

2.2 Renormalisation and BPHZ characters

Renormalisation is encoded as an action of a character g on the Hopf algebra of decorated trees. Divergent subdiagrams (trees with nonintegrable kernels) produce divergent expectations of model evaluations as the mollifier scale $\varepsilon \downarrow 0$. The BPHZ procedure prescribes a subtraction character g^ε which removes these diverging pieces at the algebraic level; applying g^ε to the canonical model yields a renormalised model with uniform bounds. At the PDE level, the effect is identical to adding counterterms (linear and nonlinear) whose coefficients are given by the same combinatorial assignments.

2.3 Vector indices and gauge structure

For Maxwell-type systems -of interest to us- the symbol space must carry vector indices and allow for index-contractions corresponding to inner products, curls and divergences. This increases combinatorial complexity: trees now carry index labels and contraction rules. Gauge symmetry (e.g. invariance under $A \mapsto A + \nabla\phi$) constrains admissible counterterms; gauge-fixing (Coulomb gauge or penalisation) is used to place the problem in an elliptic/parabolic regime and to isolate transverse vs longitudinal modes. In practice, one introduces separate symbol types for longitudinal and transverse components and tracks how renormalisation affects each.

2.4 Typical divergent diagrams

The simplest divergent diagrams in low dimensions are the tadpole (single-loop) and double-contraction diagrams. For a cubic nonlinearity these produce a linear mass-type counterterm (proportional to A) and a constant shift. For vector fields index contractions can make the coefficients different for transverse and longitudinal projections; hence counterterms such as $C_1 A$ and $C_2\nabla(\nabla \cdot A)$ may appear.

2.5 Numerical evaluation of counterterms

Numerically, counterterms are computed by evaluating diagrammatic integrals of products of Green's functions and mollifiers. On a lattice, this reduces to discrete sums in Fourier space which can be accelerated by FFTs. Careful extrapolation $h \to 0$ (mesh refinement) and monitoring of mollifier-dependence allows verification of predicted divergence rates (e.g. logarithmic or power-law) and extraction of finite renormalised values under chosen subtraction schemes.

3 Model SPDEs

We study the following family of stochastic Maxwell-type PDEs on the torus $\mathbb{T}^d$ ($d \le 3$):

$$\partial_t A(t,x) + \gamma A(t,x) + \operatorname{curl}\operatorname{curl} A(t,x) + \lambda_1 (A\cdot\nabla)A + \lambda_2 |A|^2 A = \Xi(t,x), \tag{3.1}$$

with initial data $A(0,\cdot) = A_0$. Here Ξ is a centered Gaussian noise with covariance

$$\mathbb{E}[\Xi_i(t,x)\Xi_j(s,y)] = \delta_{ij}\,\rho_\varepsilon(t-s, x-y), \tag{3.2}$$

where ρ_ε is a space-time mollifier of scale $\varepsilon > 0$. We impose the Coulomb gauge $\nabla \cdot A = 0$, or incorporate a gauge-fixing penalty to control divergences:

$$\mathcal{L}_\kappa A := \operatorname{curl}\operatorname{curl} A - \kappa\nabla(\nabla\cdot A), \tag{3.3}$$

so that $\mathcal{L}_\kappa$ is elliptic for $\kappa > 0$.

Assumption 3.1 (Subcriticality). *Assume the noise regularity and nonlinearity degrees are such that the equation* (3.1) *is subcritical in the sense of regularity structures: formally, the scaling dimension* $|\Xi|$ *yields a finite set of decorated trees of negative homogeneity. Concretely for* $d \le 3$ *with space-time parabolic scaling* $(2, 1, \dots, 1)$ *and white-in-time, spatially homogeneous noise mollified at scale* ε*, the quartic nonlinearity* $|A|^2 A$ *is subcritical.*

4 Regularity structure construction for vector photon fields

We now outline the construction of a regularity structure $(\mathcal{T}, \mathcal{G})$ adapted to the system (3.1). The construction mirrors scalar cases but must track vector indices and divergence/curl operations.

4.1 Symbols and homogeneities

Let Ξ_i denote the noise symbol for component $i \in \{1,2,3\}$. Introduce integration map $\mathcal{I}$ corresponding to the linear operator inverse K of $\partial_t + \gamma + \mathcal{L}_\kappa$ (the Green's operator). We define:

$$\{\mathbf{1}, X^k, \Xi_i,\ \mathcal{I}[\tau],\ \tau_1 \cdot \tau_2,\ \nabla \cdot \tau,\ \operatorname{curl} \tau\},$$

where X^k are polynomial symbols and products account for nonlinearities. Each symbol carries a homogeneity $|\cdot|$ defined by:

$$\begin{aligned} &|\Xi| = \alpha < 0, \qquad |X^k| = |k|, \qquad |\mathcal{I}[\tau]| = |\tau| + 2, \\ &|\tau_1 \cdot \tau_2| = |\tau_1| + |\tau_2|. \end{aligned} \tag{4.1}$$

We carry our vector indices: Ξ_i transforms under SO(3) as a vector, and multilinear products carry tensor index structure.

4.2 Model space and an interesting structure group

The model space $\mathcal{T} = \bigoplus_{\beta \in A} \mathcal{T}_\beta$ is graded by homogeneities $A \subset \mathbb{R}$ finite below any fixed level. Each $\mathcal{T}_\beta$ is a finite-dimensional vector space spanned by basis "decorated" trees (vector-labelled). The structure group $\mathcal{G}$ acts by translation of basepoint and renormalisation characters. Its definition requires combinatorial coproducts adapted to vector index contractions and differential operators ∇, curl.

4.3 Models and admissible kernels

Given the mollified noise $\Xi^\varepsilon = \rho_\varepsilon * \Xi$, define a canonical model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ by:

$$\Pi_x^\varepsilon(\Xi_i)(\varphi) := \langle \Xi_i^\varepsilon, \varphi \rangle, \qquad \Pi_x^\varepsilon(\mathcal{I}[\tau]) := K * \Pi_x^\varepsilon(\tau),$$

and extend multiplicatively. Admissibility requires K to satisfy the required kernel bounds (regularising of order 2) and vectorial mapping properties (boundedness with respect to curl/divergence), analogues of assumptions in standard constructions.

5 Renormalisation and main theorem

The canonical models Z^ε diverge as $\varepsilon \to 0$. We renormalise using a Hopf-algebraic / BPHZ scheme adapted to vector symbols.

5.1 Renormalisation group and counterterms

Let $\mathcal{R}$ denote the renormalisation group of characters $g : \mathcal{T} \to \mathbb{R}$ preserving polynomials. For each ε there exists $g^\varepsilon \in \mathcal{R}$ such that the renormalised model

$$\widehat{Z}^\varepsilon := g^\varepsilon Z^\varepsilon$$

converges as $\varepsilon \downarrow 0$ to a limiting model $\widehat{Z}$. The renormalisation character g^ε is determined by the divergent subdiagrams (trees) and yields explicit counterterms in the equation. For (3.1) the renormalised equation takes the form:

$$\partial_t A + \gamma A + \mathcal{L}_\kappa A + \mathcal{N}(A) + C_1^\varepsilon A + C_2^\varepsilon \nabla(\nabla \cdot A) + C_3^\varepsilon J(A) = \Xi^\varepsilon, \tag{5.1}$$

where $C_1^\varepsilon, C_2^\varepsilon, C_3^\varepsilon$ are mollifier-dependent scalar renormalisation constants and $J(A)$ denotes a lower-order current-type counterterm (e.g. cubic in A) arising from contraction of index-carrying diagrams. The limits $C_j^\varepsilon \to \infty$ as $\varepsilon \to 0$ in a prescribed way, but the renormalised dynamics obtained after subtracting these terms is finite.

5.2 Main theorem (local well-posedness)

Theorem 5.1 (Local well-posedness of renormalised Maxwell SPDE)**.** *Under Assumption 3.1 and for initial data A_0 in a suitable Besov/Hölder space $\mathcal{C}^\eta$ (with η above the negative homogeneities threshold), there exist renormalisation characters g^ε and counterterms C_j^ε such that:*

1. *the renormalised models $\widehat{Z}^\varepsilon = g^\varepsilon Z^\varepsilon$ converge in probability to a limiting admissible model $\widehat{Z}$ as $\varepsilon \downarrow 0$;*
2. *for each ε, the renormalised equation (5.1) admits a unique local-in-time solution A^ε which can be represented as a modelled distribution with respect to $\widehat{Z}^\varepsilon$;*
3. *the reconstructions $\mathcal{R}^{\widehat{Z}^\varepsilon} A^\varepsilon$ converge in probability in $\mathcal{C}^\alpha_{\mathrm{loc}}$ to a limit A as $\varepsilon \downarrow 0$; the limit A is called the solution of the renormalised Maxwell SPDE and depends continuously on the initial data and the renormalisation scheme up to natural finite renormalisation ambiguities.*

Remark 5.2. *The theorem is the Maxwell-vector analogue of local well-posedness results for parabolic singular SPDEs. Gauge-fixing (choice of κ) and damping γ play essential roles in ensuring parabolic regularisation and subcriticality; without them the problem is significantly harder (hyperbolic regime).*

6 Proof sketch

We summarise the main steps, following the paradigm of regularity structures.

1. Model space and canonical model. Construct the graded symbol space and define the canonical model Z^ε for the mollified noise as in Section 4. Verify analytic bounds (model estimates) up to stochastic moments; divergences manifest as $\varepsilon \to 0$ in expectations of certain model evaluations.

2. Renormalisation (BPHZ procedure). Identify divergent subtrees (those with negative homogeneity and nonintegrable kernels). Compute their expectations as functions of ε. Define g^ε by subtracting those expectations (character on the Hopf algebra). This yields finite renormalised model $\widehat{Z}^\varepsilon$ with uniform (in ε) model bounds.

3. Fixed point in modelled distributions. Reformulate (3.1) as a fixed-point of a map in the space $\mathcal{D}^\gamma$ of modelled distributions relative to $\widehat{Z}^\varepsilon$:

$$U = \mathcal{K}(\mathcal{R}U \mapsto \mathcal{N}(\mathcal{R}U) + \Xi^\varepsilon) + \mathrm{lift}(A_0), \tag{6.1}$$

where $\mathcal{K}$ is the abstract integration operator corresponding to K. Use Schauder estimates in the regularity structure to prove contraction on a small time interval. The renormalisation counterterms appear as necessary modifications of $\mathcal{N}$ at the abstract level.

4. Reconstruction and convergence. Apply the Reconstruction Theorem to obtain classical distributions from modelled distributions. Use compactness and uniform bounds on $\widehat{Z}^\varepsilon$ and solutions to pass $\varepsilon \downarrow 0$, obtaining the limit solution.

7 Example: simplified (toy) model in $d = 2$

Consider a scalarized toy version (one vector component) with cubic nonlinearity on $\mathbb{T}^2$:

$$\partial_t u + \gamma u - \Delta u + u^3 = \Xi. \tag{7.1}$$

This is analogous to the Φ^4_2 model but with time derivative and damping. The regularity structure is the usual one for Φ^4 with vector decorations suppressed. Divergent diagrams are the tadpole and the double-contraction; renormalisation yields counterterms:

$$u^{(\mathrm{note})} u^3 \mapsto u^3 - 3C^\varepsilon_{\mathrm{tad}} u - C^\varepsilon_{\mathrm{const}}.$$

Numerical evaluation of $C^\varepsilon_{\mathrm{tad}}$ and $C^\varepsilon_{\mathrm{const}}$ proceeds via Monte Carlo integrals of the mollified kernel:

$$C^\varepsilon_{\mathrm{tad}} = \int K_\varepsilon(0,y)^2\,dy, \qquad C^\varepsilon_{\mathrm{const}} = \int K_\varepsilon(0,y)^3\,dy, \tag{7.2}$$

where K_ε is the mollified Green's function.

8 Numerical considerations: discretization and renormalisation constants

In theory, to compute C^ε_j numerically one may:

1. discretize space-time on a lattice of mesh $h \ll 1$, replace Ξ^ε by lattice white noise convolved with a discrete mollifier ρ^h_ε;
2. approximate Green's function K^h (discrete inverse of $\partial_t + \gamma + \mathcal{L}_\kappa$) in Fourier domain;
3. evaluate diagram integrals as discrete sums; e.g.

$$C^{\varepsilon,h}_{\mathrm{tad}} \approx h^d \sum_k \big(K^h(k)\widehat{\rho^h_\varepsilon}(k)\big)^2.$$

4. extrapolate $h \to 0$ for fixed ε and then $\varepsilon \to 0$ with subtraction scheme to validate divergence rates $\sim \log \varepsilon$ or ε^{-1} depending on dimension.

One can accelerate computations using FFTs and variance reduction for Monte Carlo expectations. The discrete counterterms provide the subtraction constants needed in simulated renormalised equations. These computations may be of interest and serve future research.

9 Discussion

This work provides a principled route to treat singular stochastic Maxwell-type equations with nonlinear interactions. Key observations:

- Damping and gauge-fixing are instrumental in placing the problem into a (parabolic) subcritical regime accessible to regularity structures.
- Vector index structures increase combinatorial complexity but are handled by decorated trees carrying index contractions; renormalisation constants may depend on index contractions (hence different counterterms for longitudinal vs transverse modes).
- Numerical computation of renormalisation constants is feasible via lattice approximations; this is essential for connecting theory to simulations of interacting photon fields.

10 Open problems and future directions

1. Extend analysis to truly hyperbolic Maxwell equations (removing damping) — requires adaptation of regularity structures to hyperbolic scaling.
2. Incorporate coupling to quantum matter fields (e.g. stochastic Dirac fields) in the regularity structures framework.
3. Rigorous derivation of renormalisation group flow for counterterms as physical cutoff is removed.